Characteristic Modes
Part I: Introduction

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Characteristic Modes

Conventionally, characteristic modes $I_n$ are defined as

$$XI_n = \lambda_n RI_n,$$

in which $Z = R + jX$ is the impedance matrix.

Dominant characteristic mode of helicopter model discretized into 18989 basis functions, $ka = 1/2$, decomposed into characteristic modes in AToM in 47s.
Characteristic Modes

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However, who knows:

- What is impedance matrix and how to get it?
- What the hell are the characteristic modes?
- Why are they of our interest?

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...the characteristic mode theory is to be systematically derived.
Characteristic Modes

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- What the hell are the characteristic modes?
- Why are they of our interest?

Therefore, . . .

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Disclaimer: There will be equations! Brace yourself and be prepared...
This talk concerns:

- electric currents in vacuum (generalization is, however, straightforward),
- time-harmonic quantities, i.e., \( \mathcal{A}(r,t) = \text{Re} \{ A(r) \exp(j\omega t) \} \).
Electric Field Integral Equation\(^1\)

\[ \int_{\Omega} \sigma \rightarrow \infty \quad \text{(PEC)} \]

Original problem.


\begin{equation*}
\hat{n} \times (E_{s}(r') + E_{i}(r')) = 0, \quad r' \in \Omega
\end{equation*}

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Electric Field Integral Equation\textsuperscript{1}

\[ \hat{n} \times (E_S(r') + E_i(r')) = 0, \quad r' \in \Omega \]

Necessary Background

Electric Field Integral Equation\(^1\)

\[
\mathbf{k} \cdot \mathbf{E}_s (\mathbf{r}) \\
\sigma \rightarrow \infty \\
(\text{PEC}) \\
\mathbf{k} \cdot \mathbf{E}_i (\mathbf{r})
\]

Original problem.

\[
\hat{n} \times \left( \mathbf{E}_s (\mathbf{r}') + \mathbf{E}_i (\mathbf{r}') \right) = 0, \quad \mathbf{r}' \in \Omega
\]

\[
-\hat{n} \times \mathbf{E}_i (\mathbf{r}') = \mathbf{Z} (\mathbf{J}), \quad \mathbf{J} = \mathbf{J} (\mathbf{r}')
\]

Equivalent problem.

Electric Field Integral Equation – Problem Formalization

Key role of the impedance operator $\mathcal{Z}(J)$

$$\hat{n} \times \hat{n} \times \mathbf{E}_s(r') = \mathcal{Z}(J) = -\hat{n} \times \hat{n} \times (j\omega \mathbf{A} + \nabla \varphi).$$

Substituting for Lorenz gauge-calibrated potentials\(^2\) $\mathbf{A}$ and $\varphi$ gives

$$\mathcal{Z}(J) = jkZ_0 \int_{\Omega} \mathbf{G}(r,r') \cdot \mathbf{J}(r') \, dS$$

Necessary Background

Electric Field Integral Equation – Problem Formalization

Key role of the impedance operator $\mathcal{Z}(J)$

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Substituting for Lorenz gauge-calibrated potentials\(^2\) $A$ and $\varphi$ gives

$$\mathcal{Z}(J) = jkZ_0 \int_{\Omega} G(r,r') \cdot J(r') \, dS = jkZ_0 \int_{\Omega} \left( 1 + \frac{1}{k^2} \nabla \nabla \right) \cdot J(r') \frac{e^{-jk|r'-r|}}{4\pi |r'-r|} \, dS,$$

- Impedance operator $\mathcal{Z}$ is linear, symmetric (reciprocal, thus non-Hermitian).
- Alternative formulation MFIE\(^3\), common extension towards CFIE\(^3\).

---


Only canonical bodies can typically be evaluated analytically.

Problem \(-\hat{n} \times \hat{n} \times E_i (r') = Z(J)\) has to be solved numerically!

Equivalent problem.

---

Dicretization of the Problem

Only canonical bodies can typically be evaluated analytically. Problem $-\hat{n} \times \hat{n} \times \mathbf{E}_i (r') = \mathbf{Z} (J)$ has to be solved numerically!

- Discretization \( \Omega \rightarrow \Omega_T \) is needed (nontrivial task!)

Engineers like linear systems

\[ \mathcal{L}(f) = h. \]

Typically unsolvable for \( f \) in the present state (how to invert \( \mathcal{L} \))?.
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- Typically unsolvable for \( f \) in the present state (how to invert \( \mathcal{L} \)?)

Representation in a basis \( \{\psi_n\} \) and linearity of operator \( \mathcal{L} \) readily gives\(^5\)

\[
\sum_{n=1}^{N} I_n \mathcal{L}(\psi_n) = h.
\]

- One equation for \( N \) unknowns \( \rightarrow \) still unsolvable.

Using proper inner product $\langle \cdot, \cdot \rangle$ and $N$ tests from left, we get

\[
\sum_{n=1}^{N} I_n \langle \chi_n, \mathcal{L} (\psi_n) \rangle = \langle \chi_n, h \rangle,
\]

\textit{i.e.}, in matrix form the method of moments\textsuperscript{5} relation reads

\[ LI = H. \]

Algebraic Solution – Method of Moments

Piecewise basis functions\(^6\)

\[
\psi_n(r) = \frac{l_n}{2A_n} \rho^\pm(r)
\]


RWG basis function \(\psi_n\).
Piecewise basis functions\(^6\)

\[
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are applied to approximate \(J(r)\) as

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Galerkin testing\(^7\), i.e., \( \{ \chi_n \} = \{ \psi_n \} \), is performed as

\[ \int_\Omega \psi \cdot \mathcal{Z}(\psi) \, dS = \langle \psi, \mathcal{Z}(\psi) \rangle \equiv \mathbf{Z} = [Z_{pq}] \in \mathbb{C}^{N \times N}. \]


\(^7\)P. M. Morse and H. Feshbach, Methods of Theoretical Physics. McGraw-Hill, 1953
From Impedance Operator $Z$ to Impedance Matrix $Z$

The impedance matrix $Z$ reads

$$Z_{pq} = \int_{\Omega} \psi_p \cdot Z(\psi_q) \, dS = jkZ_0 \int_{\Omega} \int_{\Omega} \psi_p(r_1) \cdot G(r_1, r_2) \cdot \psi_q(r_2) \, dS_1 \, dS_2.$$
From Impedance Operator $\mathcal{Z}$ to Impedance Matrix $\mathbf{Z}$

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We say\(^8\) “Matrix $\mathbf{Z}$ is the impedance operator $\mathcal{Z}$ represented in $\{\psi_n\}$ basis.”

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- We say$^8$: “Matrix $Z$ is the impedance operator $\mathcal{Z}$ represented in $\{\psi_n\}$ basis.”
- Matrix $Z$ can be calculated, e.g., in AToM$^9$ (plenty of numerical techniques and tricks should/can be used$^{10}$).

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From Impedance Operator \( \mathcal{Z} \) to Impedance Matrix \( Z \)

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- We say\(^8\): “Matrix \( Z \) is the impedance operator \( \mathcal{Z} \) represented in \( \{\psi_n\} \) basis.”
- Matrix \( Z \) can be calculated, \( e.g. \), in AToM\(^9\) (plenty of numerical techniques and tricks should/can be used\(^{10}\)).
- Generally\(^{11}\), impedance matrix \( Z \) inherits properties of impedance operator \( \mathcal{Z} \).
  - Symmetric, complex-valued.

---


From Impedance Operator $\mathcal{Z}$ to Impedance Matrix $\mathbf{Z}$

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$$

- We say\textsuperscript{8}: “Matrix $\mathbf{Z}$ is the impedance operator $\mathcal{Z}$ represented in $\{\psi_n\}$ basis.”
- Matrix $\mathbf{Z}$ can be calculated, \textit{e.g.}, in AToM\textsuperscript{9} (plenty of numerical techniques and tricks should/can be used\textsuperscript{10}).
- Generally\textsuperscript{11}, impedance matrix $\mathbf{Z}$ inherits properties of impedance operator $\mathcal{Z}$.
  - Symmetric, complex-valued.
- Matrix $\mathbf{Z}$ completely describe the scattering properties of radiator $\Omega_T$.

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\textsuperscript{9} (2017). Antenna Toolbox for MATLAB (AToM), Czech Technical University in Prague, [Online]. Available: \url{www.antennatoolbox.com}


## Two Hilbert Space Representations

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<td>bilinear form (for $\mathcal{Z}$)</td>
<td>$p = \langle J, \mathcal{Z}(J) \rangle$</td>
<td>$p \approx I^H ZI$</td>
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<td></td>
<td>$\langle f, g \rangle = \int_{\Omega} f^*(x) \cdot g(x) , dx$,</td>
<td>$A^H = (A^T)^*$</td>
</tr>
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</table>

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Example: Complex Power Balance

Quantity being important in the following.
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▶ Continuous form\textsuperscript{13} using operator $\mathcal{Z}$

\[
-\frac{1}{2} \int_{\Omega} J^* \cdot E_s \, dS = \frac{1}{2} \int_{\Omega} J^* \cdot \mathcal{Z}(J) \, dS = P_{rad} + 2j\omega(W_m - W_e).
\]

Example: Complex Power Balance

Quantity being important in the following.

- Continuous form\(^{13}\) using operator \(Z\)

\[
\frac{1}{2} \int_\Omega J^* \cdot E_s \, dS = \frac{1}{2} \int_\Omega J^* \cdot Z(J) \, dS = P_{\text{rad}} + 2j\omega (W_m - W_e).
\]

- Algebraic form\(^{14}\) using matrix \(Z\)

\[
\frac{1}{2} \int_\Omega J^* \cdot Z(J) \, dS \approx \frac{1}{2} I^H Z I = P_{\text{rad}} + 2j\omega (W_m - W_e).
\]


Motivation

Describe behavior of a scatterer $\Omega$ without feeding considered.
Definition of Characteristic Modes

Diagonalization of the Impedance Operator/Matrix

Motivation
Describe behavior of a scatterer $\Omega$ without feeding considered.

Diagonalization of the impedance matrix $Z$:

$$
\begin{bmatrix}
Z_{11} & Z_{12} & Z_{13} & \cdots & Z_{1N} \\
Z_{21} & Z_{22} & Z_{23} & \cdots & Z_{2N} \\
Z_{31} & Z_{32} & Z_{33} & \cdots & Z_{3N} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
Z_{N1} & Z_{N2} & Z_{N3} & \cdots & Z_{NN}
\end{bmatrix}
\begin{bmatrix}
\nu_1 & 0 & 0 & \cdots & 0 \\
0 & \nu_2 & 0 & \cdots & 0 \\
0 & 0 & \nu_3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \nu_N
\end{bmatrix}
$$

Impedance matrix $Z$. Yet-unknown diagonalization of impedance matrix $Z$. 
Impedance Operator Represented in Spherical Harmonics

Example: Let us represent the impedance operator $\mathcal{Z}$ in a basis\textsuperscript{15} of spherical harmonics $\{J_{n}^{\text{sh}}(\vartheta, \varphi, a)\} \in \mathbb{R}$

\[
\langle J_{m}^{\text{sh}}, \mathcal{Z}(J_{n}^{\text{sh}}) \rangle = \int_{\Omega} J_{m}^{\text{sh}} \cdot \mathcal{Z}(J_{n}^{\text{sh}}) \, dS,
\]

Spherical shell of radius $a$.

\textsuperscript{15} This can be understood as solving method of moments analytically with spherical harmonics as basis functions.

Example: Let us represent the impedance operator $\mathcal{Z}$ in a basis\(^{15}\) of spherical harmonics $\{J_{n}^{\text{sh}}(\vartheta, \varphi, a)\} \in \mathbb{R}$

$$\langle J_{m}^{\text{sh}}, \mathcal{Z} \left( J_{n}^{\text{sh}} \right) \rangle = \int_{\Omega} J_{m}^{\text{sh}} \cdot \mathcal{Z} \left( J_{n}^{\text{sh}} \right) \, dS,$$

This representation gives diagonal matrix\(^{16}\), i.e.,

$$\langle J_{m}^{\text{sh}}, \mathcal{Z} \left( J_{n}^{\text{sh}} \right) \rangle = 2 \left( P_{\text{rad},n}^{\text{sh}} + 2j\omega \left( W_{m,n}^{\text{sh},e} - W_{m,n}^{\text{sh},e} \right) \right) \delta_{mn},$$

$$\delta_{mn} = 1 \iff m = n \quad \wedge \quad \delta_{mn} = 0 \iff m \neq n.$$
Characteristic Modes of Spherical Shell

Formula for spherical shell normalized to unitary radiated power (no units)

$$\frac{\langle J^n_{sh,m}, Z(J^n_{sh,n}) \rangle}{\langle J^n_{sh,m}, R(J^n_{sh,n}) \rangle} = \left( 1 + j \frac{2\omega(W^n_{sh,m,n} - W^n_{sh,e,n})}{P^n_{sh,rad,n}} \right) \delta_{mn},$$

where $Z = R + j\mathcal{X}$
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\frac{\langle J_{m}^{\text{sh}}, \mathcal{Z} (J_{n}^{\text{sh}}) \rangle}{\langle J_{m}^{\text{sh}}, \mathcal{R} (J_{n}^{\text{sh}}) \rangle} = \left( 1 + j \frac{2\omega (W_{m,n}^{\text{sh}} - W_{e,n}^{\text{sh}})}{P_{\text{sh rad},n}} \right) \delta_{mn},
\]

where \( \mathcal{Z} = \mathcal{R} + j\mathcal{X} \) or, alternatively (without problems due to division by zero),

\[
\langle J_{m}^{\text{sh}}, \mathcal{R} (J_{n}^{\text{sh}}) + j\mathcal{X} (J_{n}^{\text{sh}}) \rangle = \left( 1 + j\lambda_{n}^{\text{sh}} \right) \langle J_{m}^{\text{sh}}, \mathcal{R} (J_{n}^{\text{sh}}) \rangle \delta_{mn}.
\]
Characteristic Modes of Spherical Shell

Formula for spherical shell normalized to unitary radiated power (no units)

\[
\frac{\langle J_{sh}^m, Z(J_{sh}^n) \rangle}{\langle J_{sh}^m, R(J_{sh}^n) \rangle} = \left(1 + j\frac{2\omega (W_{sh,m,n} - W_{sh,e,n})}{P_{sh,rad,n}}\right) \delta_{mn},
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where \(Z = R + j\mathcal{X}\) or, alternatively (without problems due to division by zero),

\[
\langle J_{sh}^m, R\left(J_{sh}^n\right) + j\mathcal{X}\left(J_{sh}^n\right) \rangle = \left(1 + j\lambda_{sh}^n\right) \langle J_{sh}^m, R\left(J_{sh}^n\right) \rangle \delta_{mn}.
\]

Linearity of the impedance operator allows to write

\[
\langle J_{sh}^m, \mathcal{X}\left(J_{sh}^n\right) \rangle = \lambda_{sh}^n \langle J_{sh}^m, R\left(J_{sh}^n\right) \rangle \delta_{mn},
\]
Characteristic Modes of Spherical Shell

Formula for spherical shell normalized to unitary radiated power

\[
\frac{\langle J_{m}^{\text{sh}}, Z(J_{n}^{\text{sh}}) \rangle}{\langle J_{m}^{\text{sh}}, R(J_{n}^{\text{sh}}) \rangle} = \left(1 + j \frac{2\omega (W_{m,n}^{\text{sh}} - W_{e,n}^{\text{sh}})}{P_{\text{rad},n}^{\text{sh}}} \right) \delta_{mn},
\]

where \( Z = R + j\chi \) or, alternatively (without problems due to division by zero),

\[
\langle J_{m}^{\text{sh}}, R(J_{n}^{\text{sh}}) + j\chi(J_{n}^{\text{sh}}) \rangle = (1 + j\lambda_{n}^{\text{sh}}) \langle J_{m}^{\text{sh}}, R(J_{n}^{\text{sh}}) \rangle \delta_{mn}.
\]

Linearity of the impedance operator allows to write

\[
\langle J_{m}^{\text{sh}}, \chi(J_{n}^{\text{sh}}) \rangle = \lambda_{n}^{\text{sh}} \langle J_{m}^{\text{sh}}, R(J_{n}^{\text{sh}}) \rangle \delta_{mn},
\]

which is solved for all \( n \in \{1, \ldots, \infty\} \) via generalized eigenvalue problem (GEP)

\[
\chi(J_{n}^{\text{sh}}) = \lambda_{n}^{\text{sh}} R(J_{n}^{\text{sh}}).
\]
Characteristic modes for arbitrarily shaped body are defined as GEP

$$\chi(J_n) = \lambda_n \mathcal{R}(J_n).$$
Characteristic modes for arbitrarily shaped body are defined as GEP
\[ \mathcal{X}(J_n) = \lambda_n \mathcal{R}(J_n). \]

Algebraic form\(^{17}\) is commonly used instead
\[ X \mathbf{I}_n = \lambda_n R \mathbf{I}_n, \]

with \( Z = R + jX \) being the impedance matrix and \( \mathbf{I}_n \in \mathbb{R}^{N \times 1} \) being expansion coefficients.

\(^{17}\)Only six canonical bodies can, in principle, be solved analytically.
Characteristic modes for arbitrarily shaped body are defined as GEP

\[ \mathcal{X}(J_n) = \lambda_n \mathcal{R}(J_n). \]

Algebraic form\(^\text{17}\) is commonly used instead

\[ XI_n = \lambda_n RI_n, \]

with \( Z = R + jX \) being the impedance matrix and \( I_n \in \mathbb{R}^{N\times1} \) being expansion coefficients.

We know that GEP\(^\text{18}\) is capable to diagonalize both \( R \) and \( X \) operators\(^\text{19}\).

- Behavior solely described by the impedance operator/matrix.
- No feeding present (neither \( E_i \), nor \( V \))!

---

\(^\text{17}\)Only six canonical bodies can, in principle, be solved analytically.


\(^\text{19}\)Generally, only two operators can simultaneously be diagonalized. Separable bodies are exceptional!
Historical Overview\textsuperscript{23}

1948 First mention of diagonalization of the scattering operator by Montgomery \textit{et al.}\textsuperscript{20}

\textsuperscript{23} The literature will be closely reviewed later.

Historical Overview

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1971 Generalized by Harrington and Mautz\textsuperscript{22} for antenna problem using impedance matrix $Z$ as $ZI_n = \nu_n MI_n$, $\nu_n \equiv 1 + j\lambda_n$, $M \equiv R$. 

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Historical Overview

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▶ proposal

1968 Rigorously introduced by Garbacz\textsuperscript{21} as field/current solutions $E_n/J_n$ with orthogonal far-fields and radiating unitary power.

▶ analytical form

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▶ algebraic form

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Rayleigh quotient\(^\text{24}\) is defined as

\[
\lambda_n = \frac{\int J_n^* \cdot \mathcal{X}(J_n) \, dS}{\int J_n^* \cdot \mathcal{R}(J_n) \, dS} = \frac{2\omega (W_m,n - W_e,n)}{P_{\text{rad},n}} \approx \frac{I_n^H X I_n}{I_n^H R I_n}.
\]

Characteristic Numbers $\lambda_n$

Rayleigh quotient\(^{24}\) is defined as

$$\lambda_n = \frac{\int_{\Omega} J_n^* \cdot \mathcal{X}(J_n) \, dS}{\int_{\Omega} J_n^* \cdot \mathcal{R}(J_n) \, dS} = \frac{2\omega (W_{m,n} - W_{e,n})}{P_{\text{rad},n}} \approx \frac{I_n^H X I_n}{I_n^H R I_n}.$$  

- Notice $P_{\text{rad},n} > 0 \Rightarrow \mathbf{R} > \mathbf{0}$ is required.
- Eigenvalues $\lambda_n$ represent the stationary points\(^{25}\).


Characteristic modes can be classified as

\[ W_{m,n} > W_{e,n} \Rightarrow \lambda_n > 0 \] mode is of inductive nature,

\[ W_{m,n} < W_{e,n} \Rightarrow \lambda_n < 0 \] mode is of capacitive nature,

\[ W_{m,n} = W_{e,n} \Rightarrow \lambda_n = 0 \] mode is in resonance\(^\text{26}\).

- To get current to the resonance, let us combine modes with \( \lambda_n < 0 \) and \( \lambda_n > 0 \).

\(^\text{26}\)Resonance of a modal current impressed in vacuum is doubtful. It cannot be excited independently.
Physical Meaning of Characteristic Numbers $\lambda_n$

Characteristic modes can be classified as

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$W_{m,n} < W_{e,n} \Rightarrow \lambda_n < 0$ mode is of capacitive nature,

$W_{m,n} = W_{e,n} \Rightarrow \lambda_n = 0$ mode is in resonance$^{26}$.

- To get current to the resonance, let us combine modes with $\lambda_n < 0$ and $\lambda_n > 0$.

Knowledge in group theory$^{27}$ gives understanding of

- degeneracies,

- crossings,

- crossing avoidances.

---

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Cardinality of a Set of Characteristic Modes

For a radiator $\Omega \ni r'$ of finite extent:

\[ J \approx \omega (w_m - w_e) P_{\text{rad}} J_n \lambda_n \]

Mapping between current densities and their complex power ratios.

Set of characteristic modes is infinite, but countable. Set of all currents has higher cardinality (uncountable).
Cardinality of a Set of Characteristic Modes

For a radiator $\Omega \ni r'$ of finite extent:

\[ J_n \]

\[ \lambda_n \]

Mapping between current densities and their complex power ratios.

Characteristic modes can freely be combined as

\[ J = \sum_n \alpha_n J_n. \]
Cardinality of a Set of Characteristic Modes

For a radiator $\Omega \ni r'$ of finite extent:

$J = \sum_n \alpha_n J_n$. 

- Set of characteristic modes is infinite, but countable.
- Set of all currents has higher cardinality (uncountable).

Mapping between current densities and their complex power ratios.
Characteristic Numbers $\lambda_n$ for Spherical Shell

$\lambda_n$ vs $ka$

$W_m < W_e$

TM modes
Characteristic Eigenangles $\delta_n$

Characteristic angles $^{28}$ $\delta_n$ scale the dynamics of $\lambda_n \in (-\infty, \infty)$

$$\delta_n = 180^\circ \left( 1 - \frac{1}{\pi} \arctan (\lambda_n) \right).$$

to $\delta_n \in (90^\circ, 270^\circ)$.

---

Characteristic Eigenangles $\delta_n$

Characteristic angles\(^{28}\) $\delta_n$ scale the dynamics of $\lambda_n \in (-\infty, \infty)$

$$\delta_n = 180^\circ \left(1 - \frac{1}{\pi} \arctan (\lambda_n)\right).$$

to $\delta_n \in (90^\circ, 270^\circ)$.

Similarly as for characteristic numbers:

$\lambda_n > 0 \Rightarrow \delta_n < 180^\circ$ mode is of inductive nature,
$\lambda_n < 0 \Rightarrow \delta_n > 180^\circ$ mode is of capacitive nature,
$\lambda_n = 0 \Rightarrow \delta_n = 180^\circ$ mode is in resonance.

Characteristic Eigenangles $\delta_n$ for Spherical Shell
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\[
\begin{align*}
\delta_n & \quad 0 \quad 0.5 \quad 1 \quad 1.5 \quad 2 \quad 2.5 \quad 3 \quad 3.5 \quad 4 \quad 4.5 \quad 5 \\
100 \quad 120 \quad 140 \quad 160 \quad 180 \quad 200 \quad 220 \quad 240 \quad 260
\end{align*}
\]

$W_m < W_e$

$W_m > W_e$

$TM$ modes

$TE$ modes

$k\alpha$

$\delta_n$
While simplest canonical body, spherical shell has plenty of potential issues, e.g.,

- degenerate eigenspace\(^{29}\), \(D(l) = 2l + 1\),
- conformity of spherical surface with commonly used basis functions,
- internal resonances\(^{30}\) \((\lambda_n \to \infty)\),
- computationally demanding evaluation\(^{31}\).

Spherical shell made of 194 triangles.

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\(^{31}\) Having no junctions, spherical shell (and other closed objects) has highest possible ratio between number of basis functions (unknowns) and triangles \((3/2)\).
Characteristic Modes $J_n$ of Spherical Shell

- Dominant capacitive characteristic mode (spherical harmonic $\text{TM}_{10}$).
- Dominant inductive characteristic mode (spherical harmonic $\text{TE}_{10}$).
Orthogonality relations\textsuperscript{32}

\[
\frac{1}{2} \mathbf{I}_m^H \mathbf{Z} \mathbf{I}_n = (1 + j\lambda_n) \delta_{mn},
\]

\[
\frac{1}{2} \sqrt{\frac{\epsilon_0}{\mu_0}} \int_0^{2\pi} \int_0^{\pi} \mathbf{F}_m^* \cdot \mathbf{F}_n \sin \vartheta \, d\vartheta \, d\varphi = \delta_{mn},
\]

\textit{i.e.}, orthogonalization of modal complex power and modal far-fields.

Summation of Characteristic Modes

Summation formula

\[ J = \sum_{n} \alpha_n J_n \]
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derived using linearity of the impedance operator and orthogonality of characteristic modes as

\[ J = \sum_n \frac{\langle J_n, E_i \rangle}{\langle J_n, Z(J_n) \rangle} J_n = \sum_n \frac{V^i_n}{1 + j\lambda_n} J_n \]
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with \( V^i_n \) being modal excitation coefficient\(^{33} \) and \( M_n = 1/|1 + j\lambda_n| \) being modal significance coefficient\(^{34} \).

- Connection between “external” and “modal” worlds.


**Bonus: On the Inversion of $\mathcal{Z}$ operator**

Summation formula slightly rearranged (Dirac notation used, \textit{i.e.,} $\mathcal{L} |f\rangle = |g\rangle$)

$$|J\rangle = \sum_{n} \frac{\langle J_n|E_i \rangle}{\langle J_n|\mathcal{Z}|J_n \rangle} |J_n\rangle$$
**Bonus: On the Inversion of \( \mathcal{Z} \) operator**

Summation formula slightly rearranged (Dirac notation used, i.e., \( \mathcal{L}|f\rangle = |g\rangle \))

\[
|J\rangle = \sum_n \frac{\langle J_n|E_i \rangle}{\langle J_n|\mathcal{Z}|J_n \rangle} |J_n\rangle
\]

Let’s do some magic…

\[
|J\rangle = \sum_n \frac{|J_n\rangle \langle J_n|}{\langle J_n|\mathcal{Z}|J_n \rangle} |J_n\rangle - \hat{n} \times \hat{n} \times E_i
\]

and compare with the defining formula \( | - \hat{n} \times \hat{n} \times E_i \rangle = \mathcal{Z}|J\rangle \).
**Bonus: On the Inversion of $Z$ operator**

Summation formula slightly rearranged (Dirac notation used, i.e., $\mathcal{L}|f\rangle = |g\rangle$)

$$|J\rangle = \sum_n \frac{\langle J_n|E_i \rangle}{\langle J_n|Z|J_n \rangle} |J_n\rangle$$

Let’s do some magic…

$$|J\rangle = \sum_n \frac{|J_n\rangle \langle J_n|}{\langle J_n|Z|J_n \rangle} |J_n\rangle |\hat{n} \times \hat{n} \times E_i\rangle$$

and compare with the defining formula $|-\hat{n} \times \hat{n} \times E_i\rangle = Z|J\rangle$.

We get

$$Z^{-1} \equiv \sum_n \frac{|J_n\rangle \langle J_n|}{\langle J_n|Z|J_n \rangle}.$$  

> But, do not even try to calculate!
Good theory needs time...

Notes on Characteristic Modes

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5. Other bases\(^{40}\) exist, \(i.e., \mathbf{AI}_n = \xi_n \mathbf{BI}_n\).
6. Nowadays implemented in FEKO, CST-MWS, WIPL-D, CEM One, HFSS.


Topics Recently Solved at the Department

- Analytical properties of characteristic modes\(^\text{41}\),
- implementation of characteristic modes\(^\text{42}\),
- benchmarks of commercial and in-house solvers\(^\text{43}\),
- modal Q-factor for antennas\(^\text{44}\),
- radiation efficiency of characteristic modes\(^\text{45}\),
- minimization of Q-factor using characteristic modes\(^\text{46}\).


Ongoing Research at the Department

- Improvement of characteristic modes decomposition\(^\text{47}\).
- tracking of modal data\(^\text{48}\).
- group theory (symmetries) for tracking and problem reducing\(^\text{49}\).
- characteristic modes for MLFMA\(^\text{50}\).
- interpolation using differentiated GEP,
- characteristic modes for arrays\(^\text{51}\).


\(^{50}\) M. Masek, M. Capek, P. Hazdra, et al., “Characteristic modes of electrically small antennas in the presence of electrically large platforms”, in Progress In Electromagnetics Research Symposium, St. Petersburg, Russia: IEEE, 2017, pp. 3733–3738. DOI: 10.1109/PIERS.2017.8262407

AToM – Antenna Toolbox for MATLAB

AToM developed at the department between 2014 and 2018.

- Capable to calculate matrix $Z$ (and many other matrices),
- capable to calculate the CMs, their tracking, post-processing.

Visit antennatoolbox.com
Special Interested Group

- Established by prof. Lau, Lund University,
- 77 groups worldwide, CTU/Elmag is an active member!

Visit characteristicmodes.org

Benchaining of the CMs Decomposition

| TM/TE mode order | log_{10} |\lambda_n| |
|------------------|---------|---------|
| TM modes         |         |         |
| TE modes         |         |         |

- exact
- AToM (1)
- FEKO
- AToM (8)
- KS
- WIPL-D
- IDA
- CEM One
- CMC
- Makarov

Visit elmag.org/CMbenchmark

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Characteristic mode analysis

From Wikipedia, the free encyclopedia

**Characteristic modes (CM)** form a set of functions which, under specific boundary conditions, diagonalizes operator relating fields and induced sources. Under certain conditions, the set of the CM is unique and complete (at least theoretically) and thereby capable of describing the behavior of a studied object in full.

This article deals with characteristic mode decomposition in electromagnetics, a domain in which the CM theory has originally been proposed.

**Contents** [hide]

1. Background
2. Definition
3. Matrix formulation
4. Properties
5. Modal quantities
6. Applications and further development
7. Software
8. Alternative bases
9. References

**Background** [edit source]

CM decomposition was originally introduced as set of modes diagonalizing a scattering matrix. [10] The theory has, subsequently, been generalized by Harrington and Mautz for antennas. [14] Harrington, Mautz and their students also successively developed several other extensions of the theory. [5][6][7][8]

Even though some precursors [5] were published back in the late 1940s, the full potential of CM has remained unrecognized for an additional 40 years. The capabilities of CM were revisited in 2007 and, since then, interest in CM has dramatically increased. The subsequent boom of CM theory is reflected by

**Definition** [edit source]

For the simplicity, only the original form of the CM — formulated for perfectly electrically conducting (PEC) bodies in free space — will be treated in this.

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Visit [wikipedia.org/wiki/Characteristic_mode_analysis](http://wikipedia.org/wiki/Characteristic_mode_analysis)
Course on

**Characteristic Modes: Theory and Applications**

Aimed at postgraduate research students and industrial engineers who want to acquire deep insight into the theory and applications of characteristic modes.

Visit esoa.webs.upv.es
Summary

Characteristic modes decomposition

\[ \mathbf{XI}_n = \lambda_n \mathbf{RI}_n \]

- diagonalizes impedance matrix \( \mathbf{Z} \),
- constitutes entire domain basis \( \{ \mathbf{I}_n \} \),
- generates orthogonal far-fields \( \{ \mathbf{F}_n \} \),
- allows compact representation of the radiator \( \Omega_T \).

\[ \Omega \xrightarrow{\sigma \to \infty} (\text{PEC}) \]

\[ \epsilon_0, \mu_0 \]

\[ \Omega \quad \mathbf{J}(r') \quad \Omega_T \]

\[ \mathbf{J}_1 \quad \mathbf{J}_2 \]
Questions?

For a complete PDF presentation see cepelmag.org

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